Three-Dimensional Heat-Transfer and Ablation Disturbances in High-Speed Flows

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A unified theory of steady-state disturbances in compressible laminar or turbulent boundary-layers and their interaction with the adjacent surface material is developed on the basis of a small perturbation approach. Fourier transformation is used to treat both periodic and nonperiodic disturbances within a common framework. The important effect of the highly nonuniform flow across the boundary-layer is taken into account, as are the effects of surface heat and mass transfer, lateral pressure gradient, and upstream influenc in the disturbance field. Three illustrative applications to the prediction of pressure, skin-friction, and heat-transfer disturbances are given including comparisons with experiment: the boundary-layer along a wavy structural skin with heat transfer, ablation surface cross-hatching, and flow past a rear-facing step or suction gap.

Nomenclature

C_p = constant pressure, specific heat
h = step height
H_o, H' = basic flow and perturbation total enthalpy, respectively
k = thermal conductivity
l_u = upstream influence distance
$\dot{m}_{w_o}, \Delta\dot{m}_w$ = basic flow and perturbation surface mass loss rates, respectively
M = Mach number
M_{BW} = surface injection Mach number
p_o, p' = basic flow and perturbation pressures, respectively
$\dot{q}_{w_o}, \dot{q}_w$ = basic flow and perturbation surface heat-transfer rates, respectively
T_o, T' = basic flow and perturbation absolute static temperatures, respectively
u', v', w' = perturbation velocity components in x, y, z coordinate directions
U_o = basic undisturbed boundary layer in x-direction
y = coordinate normal to surface
y_s = sonic height in boundary-layer profile
α = $2\pi/\lambda$
δ = boundary-layer thickness
δ_f = viscous disturbance sublayer thickness
ε_o = turbulent eddy kinematic viscosity
λ = wavy surface wavelength
μ, ν = laminar coefficients of dynamic and kinematic viscosity, respectively
ρ = basic flow density
ξ = coordinate resolved perpendicular to a chosen reference direction
τ_{w_o}, τ'_w = basic flow and perturbation wall shear stress, respectively

Subscripts

e = edge of boundary layer
n = component resolved in ξ-direction
o = basic undisturbed flow
s = state within solid surface
w = flow conditions at the wall surface

Introduction

THE study of three-dimensional disturbances in high-speed boundary-layer flows and their interaction with the adjacent surface material is important in the thermophysical design of high-speed flight vehicles. Isolated surface disturbances such as roughness elements, trip wires, gaps or skin joints and surface material or ablation discontinuities are known to alter significantly the local heat-transfer characteristics in both laminar and turbulent flow. On the other hand, the effects of distributed surface irregularities are also of practical interest, such as segments of rippled surface skin, distributed injection or suction or ablation, and small compression surfaces. To illuminate the basic physical features of these interaction problems and guide the development of engineering methods of analysis, it is desirable to develop a basic theory which treats all these special cases in a unified way.

The present paper describes a new unified theoretical treatment of this general problem for the case of small disturbances. Using a perturbation approach, a theory is developed that describes arbitrary three-dimensional steady-state pressure, temperature, skin-friction and heat-transfer disturbances within a high-speed boundary layer, and the resulting thermal and ablative interactions with the adjacent surface material. Particular emphasis is placed on treating the important effect of the highly nonuniform flow across the boundary layer for either laminar or turbulent flow and the thermal disturbances in both the boundary layer and surface material. Also included are heat and mass transfer perturbations at the surface and the effects of normal pressure gradient and upstream influence in the disturbance field.

The basic analytical features of the theory are described in the next section. An important unifying aspect of the present approach is the use of Fourier transformation to treat both periodic and nonperiodic disturbances within the same framework. In this way, the solution of a variety of interesting problems can be brought within the scope of the theory, as will be shown in the third section where some specific applications will be examined in detail and compared with experiment. These include flow past a wavy structural skin with heat transfer, the phenomenon of ablation surface cross-hatching, and flow past a rear-facing step or suction gap. Finally, a number of other interesting possible applications will also be discussed briefly.

General Theoretical Analysis

Simplifying Assumptions

We consider a basic compressible boundary-layer flow of a perfect gas in the x-direction past some mean surface ($y = 0$). The Prandtl and Lewis numbers are taken to be unity. Superimposed on this flow are arbitrary steady three-dimensional disturbances in velocity (u', v', w'), pressure (p'), and temperature (T') which are sufficiently small that they may be treated by a linearized theory (transonic or hypersonic mean flows being thereby excluded). Guided by the results of hydrodynamic stability studies,[1] a unified analysis of these disturbances can be developed in the first

Presented as Paper 72-93 at the AIAA 10th Aerospace Sciences Meeting, San Diego, Calif., January 17–19, 1972; submitted February 2, 1972; revision received August 1, 1972.

Index categories: Boundary Layers and Convective Heat Transfer–Laminar; Boundary Layers and Convective Heat Transfer–Turbulent; Material Ablation.

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approximation by introducing the following simplifying assumptions. a) Following the arguments of Lighthill[2] and Benjamin,[3] the mean flow is idealized as a rotational plane parallel flow with uniform static pressure p_0 and arbitrary lateral variations of density $\rho_0(y)$, velocity $U_0(y)$, Mach number $M_0(y)$, and temperature $T_0(y)$. This approximation is acceptable as long as the flow is not hypersonic.[1,4] b) Under the high Reynolds number flow conditions of practical interest, the disturbance field may be resolved into inviscid and viscous components, with the viscous component being significant only in a relatively thin sublayer near the surface.[2,3,5] Again, this tends to break down in hypersonic flow.[1,4] c) The disturbance motion within the aforementioned viscous disturbance sublayer is incompressible with negligible viscous dissipation heating effects, an approximation that several studies[1,4,6] have shown to be acceptable up to moderate supersonic Mach numbers, provided the compressibility of the mean flow in the sublayer is taken into account by evaluating properties at the wall temperature. d) In the case of turbulent flow, the correlations between the disturbance field and the turbulent fluctuations are assumed negligible, thereby permitting the turbulent case to be treated as a quasi-laminar mean flow with an appropriate eddy-viscosity model. This undoubtedly introduces some quantitative errors except for very small disturbance amplitudes,[7,8] although these may not be too large for engineering purposes.[9] e) The thermal response within the surface material, including possible ablation, is a quasi-steady one.

The Boundary-Layer Disturbance Field

Using the aforementioned assumptions in the linearized compressible Navier-Stokes equations, two sets of linear partial differential equations are obtained. The first set, which governs the inviscid part of the disturbance field, is

$$\frac{\partial^2 p'}{\partial x^2}(1 - M_0^2) + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2} - 2\left(\frac{dM_0/dy}{M_0}\right)\frac{\partial p'}{\partial y} = 0 \quad (1)$$

$$\rho_0 U_0 (\partial v'/\partial x) = -(\partial p'/\partial y) \quad (2)$$

$$\rho_0 U_0 (\partial w'/\partial x) = -(\partial p'/\partial z) \quad (3)$$

$$\rho_0 U_0 (\partial u'/\partial x) + \rho_0 (dU_0/dy) v' = -(\partial p'/\partial x) \quad (4)$$

$$U_0[\partial(C_p T' + u' U_0)/\partial x] = -(dH_0/dy) v' \quad (5)$$

Equation (1), which describes pressure perturbations in a highly nonuniform flow, has been encountered in other studies of small disturbance phenomena.[2,10,11] Note that it is uncoupled from the rest, and involves only the known Mach number profile $M_0(y)$ of the basic flow. Note further that it, together with Equation (2), permits us to include not only the effect of a lateral pressure gradient but also the upstream influence in the disturbance field. Once it is solved subject to boundary conditions at the edge of the boundary layer appropriate to the particular problem, the velocity perturbation components v', w' and u' are readily obtained by integration of Eqs. (2–4), respectively. The corresponding thermal disturbance is obtained from the total enthalpy perturbation Eqs. (5), which has evidently not been discussed heretofore in the literature. This equation shows that the total enthalpy perturbation is zero only in the special case of an adiabatic mean flow $H_0 = C_p T_0 + U_0^2/2 =$ constant, as one would expect. Moreover, it has the interesting implication that the inviscid total enthalpy disturbance is a maximum at the valleys of the perturbed streamlines.

The corresponding behavior of the disturbances near the surface where viscous and heat conduction effects are significant is governed in the first approximation by the higher-order equations

$$\frac{\partial^2}{\partial y^2}\left[(v_0 + \varepsilon_0)\frac{\partial^2 v'}{\partial y^2}\right] - \frac{\partial}{\partial x}\left[U_0 \frac{\partial^2 v'}{\partial y^2} - v'\left(\frac{d^2 U_0}{dy^2}\right)\right] = 0 \quad (6)$$

$$\frac{\partial}{\partial y}\left[(v_0 + \varepsilon_0)\frac{\partial w'}{\partial y}\right] - U_0 \frac{\partial w'}{\partial x} = \frac{1}{\rho_0}\frac{\partial p}{\partial z} \simeq -U_0\left(\frac{\partial w'}{\partial x}\right)_{\text{inviscid}} \quad (7)$$

$$\partial p'/\partial y \simeq 0 \quad (8)$$

$$(\partial u'/\partial x) + (\partial v'/\partial y) + (\partial w'/\partial z) \simeq 0 \quad (9)$$

$$(\partial/\partial y)[(v_0 + \varepsilon_0)(\partial H'/\partial y)] \simeq U_0(\partial H'/\partial x) + v'(dH_0/dy) \quad (10)$$

where v_0 and $\varepsilon_0(y)$ are the known basic flow laminar and turbulent eddy-viscosity coefficients, respectively, and where use has been made of the additional simplifying physical approximation that the dominant terms viscous and heat-conduction terms are of boundary-layer type (i.e., $d^2U'/dy^2 \gg d^2U'/dx^2$, etc.).

Equation (6) results from successive differentiation of the streamwise momentum equation using Eq. (9) to eliminate u' and w'. Note that in it we have included the profile curvature term because of its importance in the turbulent case, or when there is an axial pressure gradient in the mean flow. Thus, for example, assuming that the viscous disturbance sublayer lies within the "law of the wall" region where $\tau \simeq \tau_w$, we have

$$\rho_{0,w}[v_0 + \varepsilon_0(y)]\frac{d^2 U_0}{dy^2} \simeq \frac{dp_0}{dx} - \left(\tau_w + \frac{dp_0}{dx}y\right)\frac{d\ln(v_0 + \varepsilon_0)}{dy} \quad (11)$$

Once the basic flow is specified, Eq. (6) governing the normal perturbation velocity v' can be reduced to an ordinary fourth-order differential equation in y, following Fourier transformations with respect to x and z. The resulting solution of this equation yields the viscous displacement effect and hence the effective body shape seen by the outer inviscid disturbance field as an inner boundary condition.† The remaining velocity components w' and u' are straightforwardly obtained from Eqs. (7) and (9), as is the skin-friction disturbance $[\sim(du'/dy)(0, x, z)]$. The corresponding perturbation energy Eq. (10) can be solved in an analogous manner to obtain the temperature and, most importantly, the heat-transfer disturbance in the gas near the surface.[11]

Concerning the boundary conditions imposed on the foregoing viscous solutions, the no-slip condition $u' = w' = 0$ is, of course, required at the wall, whereas the corresponding value of v' is directly proportional to any ablative or injected mass-flow disturbance from the surface. The corresponding thermal boundary conditions for the energy equation solution depend on whether the surface is held at a fixed temperature (hence $T'_{wall} = 0$), insulated $[(\partial T'/\partial y)_{wall} = 0]$, or undergoing ablation such that the boundary-layer heat transfer and surface material mass loss rates are coupled (see below).

Response within the Surface

The three-dimensional thermal disturbance within the assumed-homogeneous surface material is governed by the heat-conduction equation perturbation

$$k_s\left(\frac{\partial^2 T_s'}{\partial x^2} + \frac{\partial^2 T_s'}{\partial y^2} + \frac{\partial^2 T_s'}{\partial z^2}\right) = \dot{m}_{w_0}\frac{\partial T_s'}{\partial y} + \Delta\dot{m}_w \frac{dT_{s_0}}{dy} \quad (12)$$

where $T_s'(x, y, z)$ is the disturbance on the mean basic temperature profile $T_{s_0}(y)$, k_s is the material conductivity (assumed constant), and $\dot{m}_{w_0}$ and $\Delta\dot{m}_w$ are the undisturbed and perturbed surface ablation rates, respectively. Note that we include the heat conduction in the streamwise direction. Depending on the specific problem, Eq. (12) can be solved by standard mathematical methods subject to the requirement that T' vanish deep within the material and the imposition of a surface boundary condition involving an interfacial energy balance between the heat transfer into the wall from the adjacent boundary layer and the heat conduction away within the wall material. For example, in the case of pure sublimation with an effective heat of ablation L_s, this condition reads

$$(\Delta\dot{m}_w)L_s + k_s(\partial T_s'/\partial y)_w \simeq [\mu_0(\partial H'/\partial y)]_w \quad (13)$$

which serves to couple the thermal perturbations in the boundary layer and surface material, respectively.

† In the special case where the mean flow is laminar or where the viscous disturbance sublayer lies within the laminar sublayer of a turbulent mean flow ($v_o \gg \varepsilon_o$, $dU_o/dy \simeq$ constant), an analytic solution involving Airy functions is possible,[3,10] including the effects of mean-flow compressibility.[11]

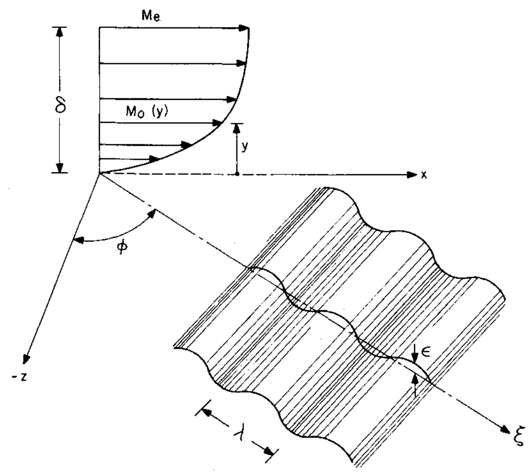

Fig. 1 Typical flow configuration (schematic).

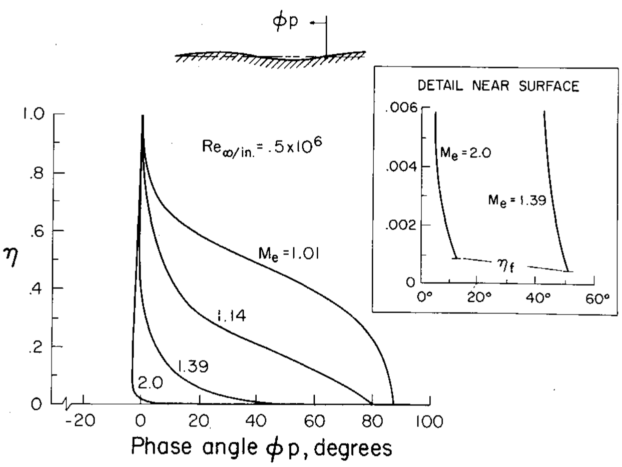

Fig. 3 Pressure phase shift across a turbulent boundary layer along a wavy wall.

Illustrative Applications of the Theory

The foregoing analysis provides a unified theoretical framework for treating a number of important heat-transfer and ablative disturbance problems. In this section we illustrate its application to three specific problems of practical interest.

Turbulent Boundary Layer on a Wavy Surface

In addition to being a classical pedagogical problem, this case finds practical application in the design of high-temperature aircraft and missile structures where the effects of skin ripples on heating are often of interest. We shall summarize here the results of a detailed theoretical and experimental study of flow past a swept slightly-wavy wall that has been made, including thermal aspects that have not been considered heretofore. A more detailed discussion of this work can be found in Refs. 11 and 12.

Excluding hypersonic flow, it is readily shown that the problem can be resolved into an equivalent two-dimensional one in a direction ξ that lies in the plane of the surface and perpendicular to the ripples (see Fig. 1). Since the disturbance field necessarily is a sinusoidal function of ξ, the general equations in the previous section may be reduced to two sets of ordinary differential equations in y. For example, in terms of the complex pressure perturbation variable $\tilde{P}(y)$ defined by $p' = \tilde{P}(y) \exp i\alpha\xi$ where $\alpha \equiv 2\pi/\lambda$, Eq. (1) reduces to

$$\{(d^2/dy^2) - 2[(dM_0/dy)/M_0](d/dy) + \alpha^2(M_{0_n}^2 - 1)\}\tilde{P} = 0 \quad (14)$$

where $M_{0_n} = M_0 \sin\phi$ is the normal Mach number and ϕ the sweep angle. Solutions of this equation have been previously studied by Lighthill[2] and more recently by Inger.[11-13] Requiring that the pressure disturbances at the boundary-layer edge be outward-running and either approach Mach waves ($M_{e_n} > 1$) or exponentially-decaying signals ($M_{e_n} < 1$), Eq. (14) can be solved numerically by a straightforward inward integration procedure. Such calculations have been carried out for the turbulent boundary-layer Mach number profiles pertaining to our wavy wall experiments. Figure 2 shows some typical results for the variation of the pressure disturbance amplitude across the boundary layer. It is seen that the nonuniform mean velocity field causes a large decrease in amplitude at low supersonic Mach numbers. It is also noted that there is evidently one Mach number (around $2^{1/2}$, see Ref. 13) where there is virtually no nonuniform flow effect. Figure 3 illustrates the corresponding pressure phase variations across the boundary layer. An appreciable shift of p_{max} toward the wall valleys (its linearized subsonic potential flow location) is seen to occur at low-supersonic Mach numbers.

Analytical solutions for the velocity and temperature disturbance fields in the viscous sublayer were also obtained. The resulting solution for the skin-friction perturbation, for example, yields

$$\tau_w' \approx C_1 R_e(p_w' e^{i\phi_c}) \quad (15)$$

where C_1 is a function of the boundary-layer thickness to wavelength ratio $\alpha\delta$, $R_e(\ )$ indicates the real part, and the phase angle ϕ_c between τ_w' and p_w' varies from 120° in an adiabatic mean flow to 60° in the case of large heat transfer. Also determined were

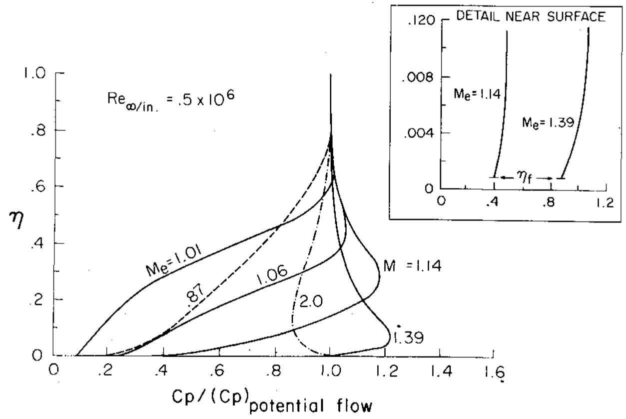

Fig. 2 Pressure perturbation amplitude variation across a turbulent boundary layer along a wavy wall.

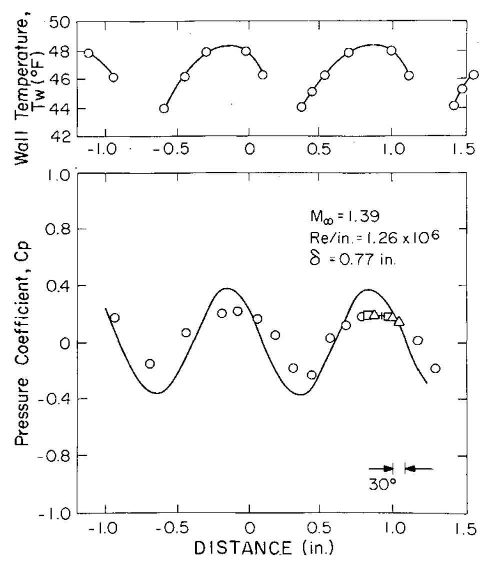

Fig. 4 Comparison of theory vs experiment for pressure and surface temperature on a wavy wall.

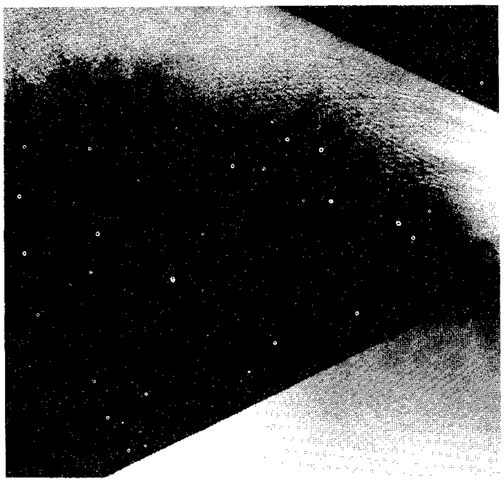

Fig. 5 Typical ablation surface cross-hatching pattern (korotherm cone at Mach 6; from Ref. 17).

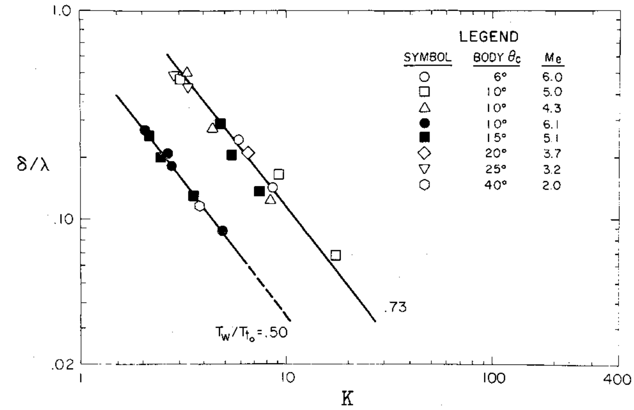

Fig. 7 Tuning parameter correlation of observed pattern wavelengths

either the heat-transfer perturbation on a wavy surface of fixed temperature, or the surface temperature perturbation for a given heat-transfer rate. For example, in the latter case (which was appropriate to our experiments), the wall temperature disturbance is found to be of the form

$$c_p T_w' \approx C_2(\alpha\delta)[-(dH_0/dy)]_w R_e[(p_w'/\tau_{w_0})]e^{i\phi_T}] \tag{16}$$

with only a small ($\phi_T = 30°$) phase difference between the maximum temperature and pressure perturbations.

Wind-tunnel experiments were carried out at low-supersonic speeds in which wall pressure and surface temperature distributions were measured along a three-cycle wavy wall in a turbulent boundary layer.[12] A typical comparison of the resulting pressure data with the predictions of the present theory is given in Fig. 4. The experiment fully substantiates two of our major theoretical conclusions, namely that a) substantial phase shifts in pressure will occur across the boundary layer, and hence b) a subsonic type of pressure signature exists on the wall ($\phi_p \equiv 90°$), even though the flow outside the boundary layer is supersonic. Another unique aspect of this experimental study, the approximate measurement of the wavy surface temperature with liquid crystal paint, is also illustrated in Fig. 4. Under our test conditions where there was a small heat transfer away from the surface ($dH_0/dy < 0$), Eq. (16) predicts a close correlation between the pressure and wall temperature perturbation maxima, and this is indeed seen to be nicely confirmed by the data.

Ablation Surface Cross-Hatching

One of the most interesting phenomena involving the mutual interaction between boundary-layer disturbances and the surface material is the occurrence of cross-hatched striation patterns on the ablated surfaces of wind-tunnel models and recovered re-entry bodies.[14-16] The typical appearance of these patterns is illustrated in Fig. 5. Since their occurrence can cause significant changes in the rolling dynamics of slender conical bodies[17,18] as well as significant increases in local heat transfer, study of their cause and growth is of practical interest.

Briefly summarized, the following major physical features have been found characterize cross-hatching[14-19]: 1) the local inviscid flow must be supersonic ($M_e > 1$); 2) the boundary layer must be transitional or turbulent; 3) the pattern sweep angle closely follows the local Mach angle based on M_e; 4) the wavelength is from two to four boundary-layer thicknesses and varies inversely with the local pressure; and 5) the patterns occur in both two-dimensional and axisymmetric flow for a wide variety of "thermally-responsive" surface materials (e.g., teflon, phenolics, camphor, wood and wax). These facts suggest that, at least in the early phases of growth, cross-hatching is caused by a self-sustaining resonant interaction between the gas dynamic and the material ablation disturbances. Such a model can be analyzed by direct application of the present theory to the case of swept disturbances which are periodic in the direction normal to the cross-hatching.[17] A detailed analysis of this case has been worked out for the particular example of a solid surface ablating strongly in pure sublimation.‡ The resulting boundary-layer disturbance solution is analogous to the foregoing wavy wall problem except that it is now intimately coupled to a corresponding spatially periodic solution of the wall thermal perturbation problem [Eqs. (12) and (13)], which is readily obtained in closed form. Further analytical details may be found in Ref. 11.

Figure 6 illustrates the most important result of the theoretical analysis: that a "resonant" interaction can indeed occur at one particular wavelength λ where the ablative mass loss in the surface valleys is a maximum. Such resonance is predicted, however, only when the material heat conduction in the ξ-direction is taken into account; otherwise, a unique wavelength does not appear. Another important result is that the theoretical solution identifies a certain "tuning" parameter K as playing a key role in this resonant interaction where

$$K = \frac{k_s/\lambda}{c_{p_s}\dot{m}_{w_0}} = \frac{\text{surface heat conduction}}{\text{ablation heat loss}} \tag{17}$$

This in turn suggests ways to suppress cross-hatching by appropriate material selection and/or distribution. It also serves as a valuable similitude parameter for correlating experimental results; for example, Fig. 7 illustrates how K serves to correlate data on pattern wavelength over a wide range of conditions.[16]

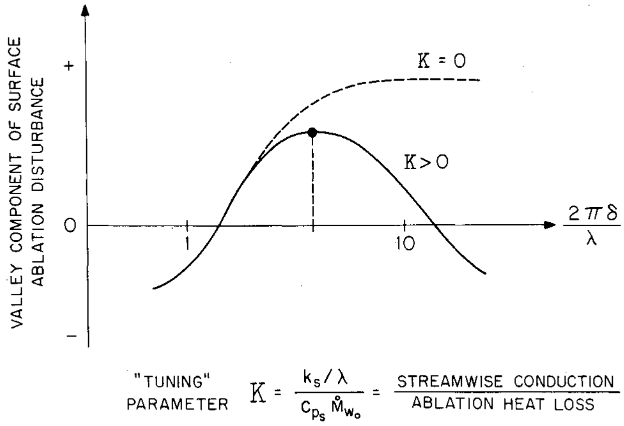

Fig. 6 Illustration of theoretically-predicted resonance between boundary layer-ablation material disturbances.

‡ Other modes of surface material response have also been considered, based on an extended version of the present theory-see, e.g., the recent paper by Grabow.[20]

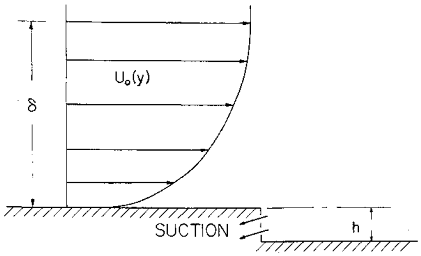

Fig. 8 Boundary-layer flow past a small step (schematic).

Supersonic Flow past a Rearward-Facing Step or Suction Gap

Although this problem has been extensively studied both theoretically and experimentally, these studies have dealt primarily with the Chapman-Korst case where the step height h is large compared to the incoming boundary-layer thickness δ, and the resulting expansion around the corner is essentially a rotational inviscid flow problem. Virtually nothing has been done for high Reynolds number flows in the opposite limit of a small step such that $h/\delta \ll 1$, as illustrated in Fig. 8. However, this latter situation is of considerable practical interest since it arises in connection with possible faults in the structural skin joints on large high-speed vehicles (such as the Space Shuttle) where the heat-transfer and pressure disturbances resulting from slight steps or gaps is of great concern.

The analysis of this small step problem falls directly within the present theoretical framework, including the effects of step sweepback and suction into the step. Thus, assuming small perturbations on the incoming nonuniform boundary-layer velocity and temperature profiles $U_0(y)$, $T_0(y)$ with $\varepsilon = h/\delta$ as the obvious small parameter, we have

$$\begin{aligned} u(x,y,z) &= U_0(y) + \varepsilon u_1'(x,y,z) + \varepsilon^2 u_2'(x,y,z) + \ldots \\ p(x,y,z) &= p_0 + \varepsilon p_1'(x,y,z) + \varepsilon p_2'(x,y,z) + \ldots \qquad (18) \\ T(x,y,z) &= T_0(y) + \varepsilon T_1'(x,y,z) + \ldots \text{etc.} \end{aligned}$$

where to first order u_1', p_1', T_1', etc. are governed by precisely the perturbation equations given in the previous section. These equations are to be solved subject to the outer boundary conditions that the viscous components of the disturbances vanish exponentially while the inviscid disturbances approach outward-running Mach waves. The inner boundary conditions must reflect the fact that the wall location drops by an amount h at the step (taken at the origin), plus allow for a possible delta function in normal perturbation velocity $\Delta v_w'$ because of suction at this origin. Consequently, if ξ is the streamwise coordinate normal to the leading edge of the step, we have the linearized wall boundary conditions at $y = 0$ so that

$$\begin{aligned} u'(\xi,0) &= 0, \ \xi \leqq 0 \\ &= (dU_0/dy)_w \cdot h, \ \xi > 0 \qquad (19) \end{aligned}$$

$$v'(\xi,0) = \Delta v_w'(\sim \text{order } h/\delta) \qquad (20)$$

plus the corresponding thermal boundary conditions that either

$$\left.\begin{aligned} T'(\xi,0) &= 0, \ \xi \leqq 0 \\ &= (dT_0/dy)_w \cdot h, \ \xi > 0 \end{aligned}\right\} \text{fixed wall temp.} \qquad (21a)$$

$$(dT'/dy)(\xi,0) = 0 \ (\text{adiabatic wall}) \qquad (21b)$$

This boundary value problem can be solved in the following manner for any step sweep angle Γ. First, we recognize that the problem may be treated as an equivalent two-dimensional one in the ξ-direction provided viscous dissipation heating effects are small; thus under the present simplifying assumptions, the governing perturbation equations can be resolved into two-dimensional form involving only the coordinates ξ and y, their respective velocity components $q' = u'\cos\Gamma + w'\sin\Gamma$ and v', and the ξ-component of the mean boundary-layer flow $q_0 \equiv U_{0_n} = U_0 \cos\Gamma$ and its corresponding Mach number M_{0_n}. Next, Fourier transformation with respect to ξ can be applied to reduce these equations to two sets of ordinary differential equations for the inviscid and viscous disturbance components which are of the same form as those already studied in the aforementioned applications [the pressure perturbation Eq. (14), for example, still applies if α therein is replaced by the Fourier frequency variable k and $\tilde{P}(y;k)$ is interpreted as the Fourier transform of p']. These transformed differential equations thus may be readily solved both analytically (for large and small k) and numerically in general subject to conditions (19) and (20), and then inverted following the method described in Refs. 2 and 11. Further details are given elsewhere.[21]

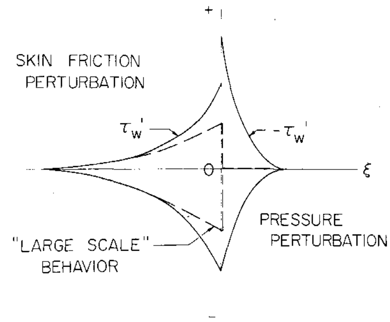

Fig. 9 Typical wall pressure and heat-transfer perturbation distributions near step with suction (schematic).

In the case where the mean velocity profile may be regarded as linear across the viscous disturbance sublayer thickness δ_f, the foregoing approach yields some very useful and instructive closed-form results for the wall pressure, skin-friction and heat-transfer perturbation distributions near the corner. The solution for the pressure, for example, assumes the following form in the leading approximation

$$p'(\xi,0) \simeq -\gamma p_0 (h/\delta_f) M_{0_n}(\delta_f)[M_{0_n}(\delta_f) - M_{B_w}]g(\xi) \qquad (22a)$$

where $M_{0_n}(\delta_f)$ indicates the mean flow boundary-layer Mach number evaluated at $y = \delta_f$, M_{B_w} is the average Mach number of injection associated with mass transfer through the step, and $g(\xi)$ is a nondimensional distribution function as follows

$$\left.\begin{aligned} g(\xi) &\simeq [M_0^2(\delta_f)/M_{0_e}^2]e^{\xi/l_u} \ (\text{large scale}) \\ &\simeq (4\delta_f/\pi y_s)e^{\xi/y_s} \ (\text{small scale}) \end{aligned}\right\} \xi < 0 \qquad (22b)$$

$$\simeq (4\delta_f/\pi y_s)e^{-3\xi/y_s}, \quad \xi > 0 \qquad (22c)$$

Here, y_s denotes the sonic height in the mean Mach number profile $[M_{0_n}(y_s) = 1]$, whereas l_u is the so-called upstream influence distance originally defined by Lighthill[10] in terms of the mean skin friction $C_{f_{n_0}}$ as

$$U_{e_n} l_u/v_e \simeq 2.87(T_w/T_e)^{5/4}(C_{f_{n_0}})^{-5/4}(\beta_{e_n})^{-3/4} \qquad (23)$$

which is related to δ_f by $\delta_f/l_u \simeq .486(C_{f_{n_0}})^{1/2}(\beta_{e_n})^{1/2}$. Equations (22) illustrate a major feature of the present theory, namely its ability to account for upstream influence in the disturbance field. In particular, Eq. (22b) shows that the upstream influence of the step is composed of two contributions: a large-scale expansion over a distance of order $\xi \sim l_u$ which is discontinuous at the step and a small-scale expansion region ($\xi < y_s$) which is continuous across the step.§ This physical behavior is illustrated qualitatively in Fig. 9. The corresponding skin friction and heat-transfer variations are found to be of the form

$$\tau_w' \approx -2[p'(\xi,0)/g(\xi)] \cdot h(\xi) \qquad (24a)$$

where

$$\left.\begin{aligned} h(\xi) &\simeq (\delta_f/l_u)[M_0^2(\delta_f)/M_{0_e}^2]e^{\xi/l_u} \ (\text{large scale}) \\ &\simeq 4/\pi(\delta_f/y_s)^2 e^{\xi/y_s} \ (\text{small scale}) \end{aligned}\right\} \xi < 0 \qquad (24b)$$

$$h(\xi) \doteq -(12/\pi)(\delta_f/y_s)^2 e^{-3\xi/y_s}, \quad \xi > 0 \qquad (24c)$$

and

$$\dot{q}_w'/\dot{q}_{w_0} \simeq \tau_w'/\tau_{w_0} \qquad (25)$$

These solutions, which are also illustrated in Fig. 9, yield the expected result that the expansion around the step intensifies the

§ Lateral pressure gradient effects become large in this region.[21]

local shear and heat transfer. Note also that they predict a discontinuous decrease across the step owing to the jump in axial pressure perturbation gradient. A comprehensive numerical study of general solutions to the present problem for the case of a nonlinear mean turbulent velocity profile, including comparisons with a related experimental study, is currently under way.

Concluding Remarks: Other Applications

The main purpose of the present paper has been to show how a number of important problems involving the thermal interaction between boundary-layer and surface disturbances can be analyzed in the first approximation within a unified theoretical framework, including the combined effects of a nonuniform laminar or turbulent boundary-layer profile, upstream influence and lateral pressure gradients in the disturbance field, compressibility and heat transfer, and either injection or ablation from the surface.

In addition to the foregoing illustrative examples, there are several other important problems that may be treated by the present approach. One is the effect of isolated roughness elements and protuberances on heat transfer and ablation. Through the use of a double general Fourier transformation with respect to x and y, plus specification of appropriate surface boundary conditions, the basic perturbation equations given herin can be rendered into sets of ordinary differential equations applicable to a variety of nonperiodic surface configurations. These equations are readily solved by numerical and analytical methods similar to those discussed previously. Another application pertains to the determination of heat transfer and skin friction in incipiently-separated boundary-layer flows. In the three applications discussed before, the basic boundary-layer flow was taken to be in zero pressure gradient; however, within the limitations of the parallel flow approximation, the present theory does not require this restriction since it treats small perturbations about any type of boundary-layer profile. Consequently, by choosing the mean boundary-layer state to be one characteristic of an adverse pressure gradient, we can study the behavior of small disturbances near separation. The ability of the present approach to treat the effects of upstream influence, lateral pressure gradient, and heat transfer would be of particular value in such a study.

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